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# A Noncommutative Generalization of the Free-Field Yang–Mills Equations

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The purpose of this paper is to propose a noncommutative generalization of a gauge connection and the free-field Yang–Mills equations. The paper draws upon the techniques proposed by Heller *et al.* for the noncommutative generalization of the Einstein field equations.

**KEY WORDS:** noncommutative geometry; gauge connections; Yang–Mills equations; groupoid algebras.

## 1. INTRODUCTION

The module of contravariant vector fields $\mathfrak{X}(M)$ on a manifold $M$ can be identified with the module of derivations of the commutative algebra $C^\infty(M)$ of smooth functions on $M$. A derivation of $C^\infty(M)$ is a mapping $X : C^\infty(M) \to C^\infty(M)$ which is linear, $X(af + bg) = aXf + bXg$, and which satisfies the so-called Leibniz rule, $X(fg) = fX(g) + X(f)g$. The set of all such derivations forms a left-module over $C^\infty(M)$; i.e., if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, then $fX \in \mathfrak{X}(M)$.

The approach of Heller *et al.* (2004) is to generalize this to a module of derivations Der$\mathcal{A}$ of a noncommutative algebra $\mathcal{A}$ of functions on a groupoid bundle. They construct a groupoid bundle $\Gamma = E \times G$ over a space–time manifold $M$, and introduce an algebra $(\mathcal{A}, *)$ of functions on $\Gamma$, where $*$ is a noncommutative convolution product. In the case where $G$ is a finite group, the module of derivations of this algebra decomposes into a direct sum of *outer* derivations and *inner* derivations, $\text{Der}\mathcal{A} = \text{Out}\mathcal{A} \oplus \text{Inn}\mathcal{A}$. The outer derivations are isomorphic to $\mathfrak{X}(M)$; they simply consist of the vector fields on $M$ lifted to $\Gamma$. The inner derivations of an algebra consist of those defined by elements of the algebra itself; each element of an algebra $f \in \mathcal{A}$ defines a derivation $X_f = ad\ f$ of $\mathcal{A}$ by the condition $X_f(g) = [f, g] = f * g - g * f$. Letting $V$ denote Der$\mathcal{A}$, Heller *et al.*

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define a generalized metric to be a mapping $\mathcal{G} : V \times V \to C^\infty(M)$, and decompose a generalized metric as the sum $\mathcal{G} = \overline{g} + h$. The first component $\overline{g}$ is the lift to $\Gamma$ of a conventional metric tensor field on $M$, $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$. The first component, therefore, provides the commutative part of a generalized metric. The second component provides the noncommutative part, $h : \text{Inn}\mathcal{A} \times \text{Inn}\mathcal{A} \to C^\infty(M)$. From the generalized metric, Heller *et al.* define a generalized Einstein tensor, which also duly decomposes into a commutative part and a noncommutative part, $G_{\overline{g}} + G_h$. Thence they define the generalized vacuum field Einstein equation to be $G_{\overline{g}} + G_h = 0$.

In the case where $G$ is a nonfinite group, Heller *et al.* (2005) employ a module of derivations with a less straightforward decomposition, $\text{Der}\mathcal{A} = \text{Der}_V\mathcal{A} \oplus \text{Der}_H\mathcal{A} \oplus \text{Inn}\mathcal{A}$, and a generalized metric with three components. Given that Heller *et al.* (2005) are interested in generalizing general relativity, they consider the case where $G$ is a noncompact semisimple Lie group, such as the Lorentz group. In contrast, this paper is interested in generalizing gauge field theory, hence it will consider the case where $G$ is a compact connected Lie group.

The approach to noncommutative gauge theory espoused in this paper differs from existing approaches which can be found in the literature. The existing approaches tend to generalize the covariant derivative rather than the connection. As Madore comments, "from the point of view of noncommutative geometry, which places primary importance upon the algebra of functions, it is [this] approach which is the more convenient and is the one which we shall consider here; the covariant derivative is defined as a linear map between modules which satisfies certain Leibniz rules. No attempt is made to define a noncommutative generalization of a connection as a 1-form on a principal fibre bundle," (1999, p. 87). In contrast, the function of this paper is to propose just such a noncommutative generalization of a connection as a 1-form.

## 2. A GENERALIZED GAUGE CONNECTION AND YANG-MILLS EQUATION

Let $M$ denote the base space–time manifold, let $G$ be a compact connected Lie group, let $E$ denote a principal $G$-bundle over $M$, and let $\Gamma$ denote the groupoid bundle over space–time, $\Gamma = E \times G$. We shall assume that $M$ is Minkowski space–time, and that $E$ is therefore a trivial bundle, isomorphic to $M \times G$. As a consequence, the groupoid bundle is isomorphic to $M \times G \times G$. Let $p$ denote an arbitrary element of $E$, let $g$ denote an arbitrary element of $G$, let $\gamma = (p, g)$ denote an element of $\Gamma$, let $x$ denote an arbitrary element of $M$, and let $\pi$ denote the projection mapping $E \to M$.

The definition of a groupoid need not detain us here (see Section 2 of Heller *et al.* (2004)). What is crucial is that the algebra of functions $\mathcal{A} = C^\infty(\Gamma, \mathrm{C})$ on the groupoid is equipped with a natural convolution product,

$$(f_1 * f_2)(\gamma) = \int_{\gamma_1 \in \Gamma_{d(\gamma)}} f_1\big(\gamma \circ \gamma_1^{-1}\big) f_2(\gamma_1)\, d\gamma_1,$$

which renders $\mathcal{A}$ a noncommutative algebra. For $\gamma = (p, g)$, $\Gamma_{d(\gamma)} \cong G$ is the set of elements $\{(p, g') : g' \in G\} = p \times G$.

Let us introduce a module of derivations of this algebra, $V = \mathrm{Der}\mathcal{A} = \mathrm{Der}_R\mathcal{A} \oplus \mathrm{Inn}\mathcal{A}$, over the ring $C^\infty(M)$. The space of derivations $\mathrm{Der}_R\mathcal{A}$ is the space of *right-invariant* vector fields on $E$, each of which lifts to a derivation of $\mathcal{A}$. As such, $\mathrm{Der}_R\mathcal{A}$ is clearly a submodule of the $C^\infty(E)$-module of vector fields on $E$, $\mathrm{Der}_R\mathcal{A} \subset \mathfrak{X}(E)$. There is, in turn, a submodule of $\mathrm{Der}_R\mathcal{A}$ consisting of vertical derivations $\mathrm{Der}_V\mathcal{A}$; these are right-invariant vector fields on $E$ which are tangent at each point $p$ to the fiber $E_x$ over $\pi(p) = x$. At each point $p \in E$ the tangent vector space $T_pE$ contains a so-called *vertical subspace* $V_pE$, consisting of the vectors which are tangent to the fiber $E_x$. A $G$-connection on $E$ selects a complementary horizontal subspace $H_p$ at each point $p$ in such a way that the horizontal subspaces are invariant under the right action of $G$:

$$H_{pg} = R_{g*}(H_p).$$

A $G$-connection on $E$ also thereby selects a complementary horizontal subspace $\mathrm{Der}_H\mathcal{A}$ so that one obtains a direct sum decomposition $\mathrm{Der}_R\mathcal{A} = \mathrm{Der}_V\mathcal{A} \oplus \mathrm{Der}_H\mathcal{A}$. Let $TE$ denote the tangent bundle of $E$, and let $T^*E$ denote the cotangent bundle. The horizontal subspace can also be defined to be the kernel of a lie-algebra valued one-form on $E$,

$$\omega : TE \to \mathfrak{g},$$

which maps the vertical subspace at each point to $\mathfrak{g}$, the lie algebra of $G$, and which respects the right action of $G$ in the sense that, for each $v \in T_pE$,

$$\omega_{pg}(R_{g*p}v) = ad(g^{-1})\omega_p(v).$$

As such, $\omega$ is a cross-section of the bundle $T^*E \otimes (E \times \mathfrak{g})$. This is the same as saying that $\omega$ is an element of $\Gamma(T^*E) \otimes \mathfrak{g}$, where $\Gamma(T^*E)$ denotes the space of cross-sections of $T^*E$. The one-form $\omega$ can also be treated as a linear mapping $\omega : \Gamma(TE) \to C^\infty(E) \otimes \mathfrak{g}$. The space of cross-sections $\Gamma(T^*E)$, is isomorphic to the dual space $\mathfrak{X}^*(E)$, hence the connection one-form $\omega$ is an element of $\mathfrak{X}^*(E) \otimes \mathfrak{g}$.

While a conventional "commutative" connection determines a decomposition of $\mathrm{Der}_R\mathcal{A}$ into $\mathrm{Der}_V\mathcal{A} \oplus \mathrm{Der}_H\mathcal{A}$, I propose that a generalized, noncommutative connection determines a decomposition of the larger space of derivations $V = \mathrm{Der}\mathcal{A} = \mathrm{Der}_R\mathcal{A} \oplus \mathrm{Inn}\mathcal{A}$ into a horizontal and a vertical subspace, $V_H \oplus V_V$.

While the horizontal subspace of a conventional connection can be defined as the kernel of a linear mapping $\omega : \Gamma(TE) \to C^\infty(E) \otimes \mathfrak{g}$, the horizontal subspace of a generalized connection would be defined as the kernel of a linear mapping

$$\mathcal{W} : V \to \mathcal{A} \otimes \mathfrak{g},$$

which maps $V_V$ to $\mathcal{A} \otimes \mathfrak{g}$, and which respects the right action of $G$ on $V$ in the sense that, for any $X \in V$,

$$\mathcal{W}(R_g X) = ad(g^{-1})\mathcal{W}(X).$$

$\mathrm{Der}_R\mathcal{A}$ is the space of vector fields on $E$ which are invariant under the right-action of $G$, hence $G$ fixes every element of $\mathrm{Der}_R\mathcal{A}$. The right action of $G$ on $\mathrm{Inn}\mathcal{A}$ is defined via the right action of $G$ on $\mathcal{A}$; $g' \in G$ maps $f \in \mathcal{A}$ to $f' \in \mathcal{A}$ where $f'(p, g) = f(pg', g)$.

Note that the generalized connection one-form is an element of $V^* \otimes \mathfrak{g}$, where $V^* = (\mathrm{Der}_R\mathcal{A})^* \oplus (\mathrm{Inn}\mathcal{A})^*$ is the set of one-forms on the derivations $V$. While the space of derivations is a left $C^\infty(M)$-module, the space of one-forms is a left $\mathcal{A}$-module. Given a basis $\{\tau_i\}$ of $(\mathrm{Der}_R\mathcal{A})^*$, and a basis $\{\theta_j\}$ of $(\mathrm{Inn}\mathcal{A})^*$, a generalized $G$-connection $\mathcal{W} : V \to \mathcal{A} \otimes \mathfrak{g}$ can be expressed as an $\mathcal{A} \otimes \mathfrak{g}$-linear combination of these basis elements:

$$\mathcal{W} = \sum_i f_i(\gamma)\tau_i + \sum_j g_j(\gamma)\theta_j$$

$f_i(\gamma)$ is the lift to $\Gamma$ of an element of $C^\infty(E) \otimes \mathfrak{g}$, while $g_j(\gamma)$ is an element of $\mathcal{A} \otimes \mathfrak{g}$. A generalized $G$-connection can be written as a sum $\mathcal{W} = \omega_1 + \omega_2$, where $\omega_1 \in (\mathrm{Der}_R\mathcal{A})^* \otimes \mathfrak{g}$ is the commutative part of the generalized connection, and $\omega_2 \in (\mathrm{Inn}\mathcal{A})^* \otimes \mathfrak{g}$ is the noncommutative part.

Given a conventional gauge connection one-form $\omega$, the conventional free-field Yang–Mills equations are

$$div\, \Omega^\omega = 0,$$

where $\Omega^\omega$ is the so-called curvature two-form of $\omega$. Given a generalized gauge connection, one can proceed analagously.

First, one can define the exterior derivative of $\mathcal{W}$ as

$$d\mathcal{W} = \sum_i df_i \wedge \tau_i + \sum_j dg_j \wedge \theta_j$$

The exterior covariant derivative of the generalized one-form $\mathcal{W}$ is defined to be the two-form $\Omega^{\mathcal{W}} : V \times V \to \mathcal{A} \otimes \mathfrak{g}$ which is such that

$$\Omega^{\mathcal{W}}(X, W) = d\mathcal{W}(X^{\mathrm{H}}, W^{\mathrm{H}}),$$

where $X$, $W$ are any pair of derivations in $V$, and $X^{\mathrm{H}}$, $W^{\mathrm{H}}$ denote the horizontal components of those derivations. The generalized two-form $\Omega^{\mathcal{W}}$ is an element of $\Lambda^2 V^* \otimes \mathfrak{g}$, where $\Lambda^2 V^*$ is the two-fold antisymmetric tensor product of $V^*$.

Hence, given a generalized $G$-connection $\mathcal{W} : V \to \mathcal{A} \otimes \mathfrak{g}$, one can define the generalized curvature two-form $\Omega^{\mathcal{W}} : V \times V \to \mathcal{A} \otimes \mathfrak{g}$. To define generalized free-field Yang–Mills equations, it is first necessary to find a generalization of the divergence operator, the contraction of the covariant derivative, $div = C \cdot \nabla$. Now, given a left $\mathcal{A}$ module $\mathcal{H}$, Madore (1999, p. 87) defines a Yang–Mills covariant derivative to be a map

$$\nabla : \mathcal{H} \to \Omega^1(\mathcal{A}) \otimes \mathcal{H},$$

where $\Omega^1(\mathcal{A})$ is the set of one-forms on the space of derivations of the algebra. In the case of relevance here, $\mathcal{H} = \Lambda^2 V^* \otimes \mathfrak{g}$, a left $\mathcal{A}$ module. Hence, a Yang–Mills covariant derivative provides a mapping:

$$\nabla : \Lambda^2 V^* \otimes \mathfrak{g} \to \Omega^1(\mathcal{A}) \otimes \Lambda^2 V^* \otimes \mathfrak{g}.$$

Given the covariant derivative $\nabla\Omega^{\mathcal{W}} \in \Omega^1(\mathcal{A}) \otimes \Lambda^2 V^* \otimes \mathfrak{g}$, to obtain the contraction one must first map one of the $V^*$-factors into a $V$-factor. To do this, one utilizes the fact that there is a natural nondegenerate metric $\mathcal{G} : V \times V \to C^\infty(M)$ on $V$ which defines a canonical isomorphism between $V$ and $V^*$. Given $V = \mathrm{Der}_V\mathcal{A} \oplus \mathrm{Der}_H\mathcal{A} \oplus \mathrm{Inn}\mathcal{A}$, one has a nondegenerate metric on each direct summand, (Heller *et al.* 2005, Section 4):

1. The trace operation $Tr$ on the lie algebra $\mathfrak{g}$ enables one to define a natural killing form metric on $\mathfrak{g}$. From this one can define a nondegenerate metric on the space of vertical right-invariant vector fields $\mathrm{Der}_V\mathcal{A}$ on $E$,

   $$g_V(Y, Z) = Tr(\iota(Y_p) \circ \iota(Z_p)).$$

   $\iota$ is an isomorphism between each vertical subspace $V_pE$ and the lie algebra $\mathfrak{g}$. Because each element of $\mathrm{Der}_V\mathcal{A}$ is right-invariant, one can take the value of such a vector field at a single arbitrary point $p$ of each fibre $E_x$, so that $g_V$ maps each pair $Y, Z \in \mathrm{Der}_V\mathcal{A}$ to an element of $C^\infty(M)$.
2. Given that each element of $\mathrm{Der}_H\mathcal{A}$ corresponds to a vector field on $M$, the metric tensor $g$ on $M$ determines a natural nondegenerate metric $g_H$ on $\mathrm{Der}_H\mathcal{A}$.
3. Finally, one can define a nondegenerate metric on $\mathrm{Inn}\mathcal{A}$ by defining $g_{\mathrm{Inn}}(ad\ a, ad\ b) = Tr(a * b)$, where

   $$(Tr\ a)(x) = \int_G a(x, g, g)\, dg,$$

   given the isomorphism between $\Gamma$ and $M \times G \times G$.

The nondegenerate metric $\mathcal{G} = g_V + g_H + g_{\mathrm{Inn}}$ on $V = \mathrm{Der}_V\mathcal{A} \oplus \mathrm{Der}_H\mathcal{A} \oplus \mathrm{Inn}\mathcal{A}$ defines a canonical isomorphism between $V$ and $V^*$ by the edict that $u^* = \mathcal{G}(u, )$. Using this isomorphism in the opposite direction, one can map one of the $V^*$-factors in $\Omega^1(\mathcal{A}) \otimes \Lambda^2 V^* \otimes \mathfrak{g}$ into a $V$-factor, and then one can contract any element of this space by allowing the $\Omega^1(\mathcal{A})$-components to act upon the $V$-components. Given the definition of the divergence $div = C \cdot \nabla$, one can define the generalized Yang–Mills equations to be

$$div\, \Omega^{\mathcal{W}} = 0.$$

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